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On the decay of solutions to the 2D Neumann exterior problem for the wave equation

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Abstract

We consider the exterior problem in the plane for the wave equation with a Neumann boundary condition and study the asymptotic behavior of the solution for large times. For possible application we are interested to show a decay estimate which does not involve weighted norms of the initial data. In the paper we prove such an estimate, by a combination of the estimate of the local energy decay and decay estimates for the free space solution.

Keywords: Wave equation; Exterior domain; Neumann boundary condition; Asymptotic behavior; Decay rate; Local energy decay

1. Introduction

Let Ω be an exterior domain in $\mathbf{R}^2$; the boundary $\partial\Omega$ is a smooth, convex and compact hypersurface. Given $r>0$, we denote $\Omega_r = \Omega \cap B_r$, where $B_r = \{x \in \mathbf{R}^2 \mid |x| < r\}$. Below, $r_0 > 0$ is a fixed constant such that $\Omega^c \subset B_{r_0}$ (Ω^c is the complement of Ω). We set $Q = [0,\infty) \times \Omega, \Sigma = [0,\infty) \times \partial\Omega$. The normal derivative on $\partial\Omega$ is denoted by ∂_ν.

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In this paper we study the decay property of solutions to the mixed problem for the wave equation with Neumann boundary condition

$$\begin{aligned} (\partial_{tt}^2 - \Delta)u = 0 &\quad \text{in } Q, \\ \partial_\nu u = 0 &\quad \text{on } \Sigma, \\ u(0,x) = f(x), & \\ \partial_t u(0,x) = g(x) &\quad \text{in } \Omega. \end{aligned} \tag{1}$$

In the previous papers [11] the first author showed the decay rate $(1+t)^{-1/2}\log^2(e+t)$, slightly slower than the optimal rate $(1+t)^{-1/2}$ of the free space solution. The aim of the present paper is to improve the dependence on the data. The result of this paper has been applied to the study of the Euler compressible flow in [12]. In view of this application, the main purpose of the present paper is to avoid the dependence on weighted norms of the data. An unpleasant consequence of this is the increase of regularity required on the data. However, we believe that this result is of own interest in itself. As in [11] our proof is a combination by a cut-off argument of the estimate of the local energy decay following from the analysis of Kleinman and Vainberg [6], Morawetz [8], Vainberg [13] and decay estimates for the free space solution. Differently from [11] we use a new estimate of the local decay of the free space solution. In order to get a decay rate of local energy in the presence of an obstacle, some assumption on its shape should be taken, in order to exclude the existence of closed ray solutions. In fact, for the Dirichlet problem, Ralston [10] has shown that if there is a closed ray solution, there is no rate of decay. For this reason we assume that the boundary is convex.

Let us introduce some notation. For a multi-index $\alpha = (\alpha_1, \alpha_2)$ we set $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}$, $|\alpha| = \alpha_1 + \alpha_2$, where $\partial_1 = \partial/\partial x_1, \partial_2 = \partial/\partial x_2$. Let $W^{m,p}(\Omega)$ be the usual Sobolev space of order m, $m = 1, 2, \dots$ and order of integrability $p \geqslant 1$, and let $||\cdot||_{W^{m,p}}$ denote its norm. If $p = 2$ we set $W^{m,p}(\Omega) = H^m(\Omega)$ with norm $||\cdot||_{H^m}$. The norm of $L^2(\Omega)$ is denoted by $||\cdot||$, the norm of $L^p(\Omega)$, $1 \leqslant p \leqslant \infty$, by $|\cdot|_p$. For simplicity we use the abbreviated notation $W^{m,p}, H^m, L^p$. We will also use the same symbol for spaces of vector valued functions.

Let us introduce the generalized derivatives

$$\partial_0 = \partial_t, \partial_1, \partial_2, \quad D = (\partial_0, \partial_1, \partial_2), \quad \omega = x_1\partial_2 - x_2\partial_1,$$

$$L_0 = t\partial_t + x_1\partial_1 + x_2\partial_2, \quad L_i = t\partial_i + x_i\partial_t \quad \text{for } i = 1, 2,$$

which we denote by $\Gamma_0, \Gamma_1, \dots, \Gamma_6$. For a multi-index $A = (A_0, A_1, \dots, A_6)$ with nonnegative integers A_i we define

$$|A| = A_0 + A_1 + \cdots + A_6, \quad \Gamma^A = \Gamma_0^{A_0}\Gamma_1^{A_1}\dots\Gamma_6^{A_6}, \quad \Gamma^0 = 1.$$

For a scalar function $u = u(t,x) : \mathbf{R}^2 \to \mathbf{R}$ and a nonnegative integer m we introduce the norm

$$|||u(t)|||_m = \max_{|A|\leqslant m}\left(\int_{\mathbf{R}^2} |\Gamma^A u(t,x)|^2\,dx\right)^{1/2} \quad \forall t\geqslant 0.$$

Let us recall the commutation relations (cf. [3,7])

$$(\partial_{tt}^2-\Delta)\Gamma_i - \Gamma_i(\partial_{tt}^2-\Delta) = 2\delta_{0i}(\partial_{tt}^2-\Delta) \quad \text{for } i=0,\dots,6,$$

$$\Gamma_i\Gamma_j - \Gamma_j\Gamma_i = \sum_{k=0}^{6} c_{ijk}\Gamma_k \quad \text{for } i,j=0,\dots,6,$$

$$\Gamma_i\partial_j - \partial_j\Gamma_i = \sum_{k=0}^{2} c^*_{ijk}\partial_k \quad \text{for } i=0,\dots,6;\ j=0,1,2$$

with certain numerical coefficients c_{ijk}, c^*_{ijk}. Because of the noncommutativity of the Γ_i one has product rules of the type

$$\Gamma^A\Gamma^B = \Gamma^{A+B} + \sum_C \gamma_{ABC}\Gamma^C \quad \text{with } |C|<|A|+|B|,$$
$$[D,\Gamma^A] = \sum_{|B|\leqslant|A|-1} \delta_{AB} D\Gamma^B = \sum_{|B|\leqslant|A|-1} \tilde{\delta}_{AB}\Gamma^B D$$

with numerical coefficients $\gamma_{ABC}, \delta_{AB}, \tilde{\delta}_{AB}$.

Theorem 1.1. *Suppose u is a solution of the exterior problem* (1). *Assume the initial data satisfy $f\in W^{6,1}$, $g\in W^{5,1}$. Then there exists a constant $C>0$ such that*

$$|\partial_t u(t,\cdot)|_\infty + |\nabla u(t,\cdot)|_\infty \leqslant C(1+t)^{-1/2}\log^2(e+t) \times (||f||_{W^{6,1}} + ||g||_{W^{5,1}}) \quad \forall t\geqslant 0. \tag{2}$$

2. Local pointwise decay

Let us consider the initial boundary value problem (1) with new notation

$$\begin{aligned} (\partial_{tt}^2-\Delta)w &= 0 \quad \text{in } Q,\\ \partial_\nu w &= 0 \quad \text{on } \Sigma,\\ w(0,x) &= w_0(x),\\ \partial_t w(0,x) &= w_1(x) \quad \text{in } \Omega. \end{aligned} \tag{3}$$

We first recall the local energy decay.

Lemma 2.1. *Let* (w_0, w_1) *have compact support and satisfy* $w_0 \in H^1, w_1 \in L^2$. *Then the solution* w *of* (3) *satisfies the estimate*

$$\int_{\Omega_r} (|w(t)|^2 + |\partial_t w(t)|^2 + |\nabla w(t)|^2)\, dx \leqslant C_r(1+t)^{-2}(||w_0||^2_{H^1} + ||w_1||^2) \tag{4}$$

for every $r > r_0$ *and* $t \geqslant 0$, *where* C_r *depends on* r, *the support of the initial data and the geometry of* $\partial\Omega$.

Proof. The decay of the local energy, i.e. of the local norm of $\partial_t w$ and ∇w, follows from [6,8,13]. Combining the results of [6,13,14] as in [1], we can include w itself in the estimate. $\square$

Lemma 2.2. *Let* (w_0, w_1) *have compact support and satisfy* $(w_0, w_1) \in H^3 \times H^2$. *Then the solution* w *of* (3) *satisfies the estimate*

$$\begin{aligned} &|w(t)|_{L^\infty(\Omega_R)} + |\partial_t w(t)|_{L^\infty(\Omega_R)} + |\nabla w(t)|_{L^\infty(\Omega_R)} \\ &\quad \leqslant C_R(1+t)^{-1}(||w_0||_{H^3} + ||w_1||_{H^2}) \end{aligned} \tag{5}$$

for every $R > r_0$ *and* $t \geqslant 0$, *where* C_R *depends on* R, *the support of the initial data and the geometry of* $\partial\Omega$.

Proof. From (3) and time differentiation, $\partial_t w$ solves

$$\begin{aligned} (\partial^2_{tt} - \Delta)\partial_t w &= 0 \quad \text{in } Q, \\ \partial_\nu \partial_t w &= 0 \quad \text{on } \Sigma, \\ \partial_t w(0,x) &= w_1(x), \\ \partial_t(\partial_t w)(0,x) &= \varDelta w_0(x) \quad \text{in } \Omega, \end{aligned} \tag{6}$$

From application of (4) to problem (6) we have

$$\int_{\Omega_r} (|\partial^2_{tt} w(t)|^2 + |\nabla \partial_t w(t)|^2)\, dx \leqslant C_r(1+t)^{-2}(||\varDelta w_0||^2 + ||w_1||^2_{H^1}),$$

which yields

$$\int_{\Omega_r} (|\varDelta w(t)|^2 + |\nabla \partial_t w(t)|^2)\, dx \leqslant C_r(1+t)^{-2}(||\varDelta w_0||^2 + ||w_1||^2_{H^1}), \tag{7}$$

for every $r > r_0$. We time differentiate once more and obtain the problem

$$\begin{aligned} (\partial^2_{tt} - \Delta)\partial^2_{tt} w &= 0 \quad \text{in } Q, \\ \partial_\nu \partial^2_{tt} w &= 0 \quad \text{on } \Sigma, \\ \partial^2_{tt} w(0,x) &= \varDelta w_0(x), \\ \partial_t(\partial^2_{tt} w)(0,x) &= \varDelta w_1(x) \quad \text{in } \Omega, \end{aligned}$$

whose solution obeys the estimate

$$\int_{\Omega_r} (|\partial^3_{ttt} w(t)|^2 + |\nabla \partial^2_{tt} w(t)|^2)\, dx \leqslant C_r(1+t)^{-2}(||\Delta w_1||^2 + ||\Delta w_0||^2_{H^1}),$$

which yields

$$\int_{\Omega_r} (|\Delta \partial_t w(t)|^2 + |\Delta \nabla w(t)|^2)\, dx \leqslant C_r(1+t)^{-2}(||\Delta w_1||^2 + ||\Delta w_0||^2_{H^1}), \tag{8}$$

for every $r > r_0$. For any fixed $t > 0$ and given $R > r_0$, we choose $\sigma(x) \in C_0^\infty(\mathbf{R}^2)$ such that $\sigma(x) = 1$ if $|x| \leqslant R$ and $= 0$ if $|x| \geqslant R+1$. Let us denote $\Phi = \Delta(\sigma \partial_t w)$. Then $\sigma \partial_t w$ solves the elliptic problem

$$\begin{aligned} \Delta(\sigma \partial_t w) &= \Phi \quad \text{in } \Omega_{R+1}, \\ \partial_\nu(\sigma \partial_t w) &= 0 \quad \text{on } \partial\Omega, \\ \sigma \partial_t w &= 0 \quad \text{on } \partial B_{R+1}. \end{aligned}$$

We then have the estimate

$$||\sigma \partial_t w||_{H^2(\Omega_{R+1})} \leqslant C ||\Phi||_{L^2(\Omega_{R+1})}. \tag{9}$$

From the Sobolev imbedding $H^2(\Omega_{R+1}) \subset L^\infty(\Omega_{R+1})$ and (9) we get

$$|\partial_t w|^2_{L^\infty(\Omega_R)} \leqslant C \int_{\Omega_{R+1}} (|\partial_t w|^2 + |\nabla \partial_t w|^2 + |\Delta \partial_t w|^2)\, dx. \tag{10}$$

From (4), (7) and (8) under the choice $r = R+1$, and (10) we then obtain

$$|\partial_t w(t)|_{L^\infty(\Omega_R)} \leqslant C_R (1+t)^{-1} (||w_0||_{H^3} + ||w_1||_{H^2}). \tag{11}$$

In order to estimate $|w(t)|_{L^\infty(\Omega_R)}$ and $|\nabla w(t)|_{L^\infty(\Omega_R)}$, we proceed similarly. In this case we consider the elliptic system

$$\begin{aligned} \Delta(\sigma w) &= \Psi \quad \text{in } \Omega_{R+1}, \\ \partial_\nu(\sigma w) &= 0 \quad \text{on } \partial\Omega, \\ \sigma w &= 0 \quad \text{on } \partial B_{R+1}, \end{aligned}$$

where we have set $\Psi = \Delta(\sigma w)$. Thus we have

$$\begin{aligned} &|w|_{L^\infty(\Omega_R)} + |\nabla w|_{L^\infty(\Omega_R)} \\ &\leqslant C ||\sigma w||_{H^3(\Omega_{R+1})} \leqslant C ||\Psi||_{H^1(\Omega_{R+1})} \\ &\leqslant C \left(\int_{\Omega_{R+1}} \left(|w|^2 + |\nabla w|^2 + \sum_{|\alpha|=2} |\partial^\alpha w|^2 + |\nabla \Delta w|^2 \right) dx \right)^{1/2}. \end{aligned} \tag{12}$$

Therefore we see the necessity to estimate all double x-derivatives of w over Ω_{R+1}. We choose $\sigma'(x) \in C_0^\infty(\mathbf{R}^2)$ such that $\sigma'(x) = 1$ if $|x| \leqslant R+1$ and $= 0$ if $|x| \geqslant R+2$. Consider the elliptic system

$$\begin{aligned} \Delta(\sigma' w) &= \Psi' \quad \text{in } \Omega_{R+2}, \\ \partial_\nu(\sigma' w) &= 0 \quad \text{on } \partial\Omega, \\ \sigma' w &= 0 \quad \text{on } \partial B_{R+2}, \end{aligned}$$

where we have set $\Psi' = \Delta(\sigma' w)$. It follows that

$$\begin{aligned} ||w||_{H^2(\Omega_{R+1})} &\leqslant ||\Delta(\sigma' w)||_{L^2(\Omega_{R+2})} \\ &\leqslant C\left(\int_{\Omega_{R+2}} (|w|^2 + |\nabla w|^2 + |\Delta w|^2)\, dx\right)^{1/2}. \end{aligned} \tag{13}$$

From (4) and (8) under the choice $r = R+1$, we obtain

$$\int_{\Omega_{R+1}} (|w|^2 + |\nabla w|^2 + |\nabla \Delta w|^2)\, dx \leqslant C_R(1+t)^{-2}(||w_0||_{H^3}^2 + ||w_1||_{H^2}^2). \tag{14}$$

From (4) and (7) under the choice $r = R+2$, and (13) we obtain

$$\sum_{|\alpha|=2} \int_{\Omega_{R+1}} |\partial^\alpha w|^2\, dx \leqslant C_R(1+t)^{-2}(||w_0||_{H^2}^2 + ||w_1||_{H^1}^2). \tag{15}$$

Finally from (12), (14) and (15) we obtain

$$|w(t)|_{L^\infty(\Omega_R)} + |\nabla w(t)|_{L^\infty(\Omega_R)} \leqslant C_R(1+t)^{-1}(||w_0||_{H^3} + ||w_1||_{H^2}). \qquad \square$$

Let us consider the initial boundary value problem

$$\begin{aligned} (\partial_{tt}^2 - \Delta)w &= G \quad \text{in } Q, \\ \partial_\nu w &= 0 \quad \text{on } \Sigma, \\ w(0,x) &= 0, \\ \partial_t w(0,x) &= 0 \quad \text{in } \Omega. \end{aligned} \tag{16}$$

Lemma 2.3. *Let $G(t,\cdot)$ have compact support independent of t and satisfy $G(t,\cdot) \in H^2$ for each $t > 0$. Then the solution w of* (16) *satisfies the estimate*

$$\begin{aligned} &|w(t)|_{L^\infty(\Omega_R)} + |\partial_t w(t)|_{L^\infty(\Omega_R)} + |\nabla w(t)|_{L^\infty(\Omega_R)} \\ &\quad \leqslant C_R \int_0^t (1+t-s)^{-1} ||G(s,\cdot)||_{H^2}\, ds \end{aligned} \tag{17}$$

for every $R > r_0$ *and* $t \geqslant 0$, *where* C_R *depends on* R, *the support of* G *and the geometry of* $\partial\Omega$.

Proof. It is a simple consequence of Duhamel's principle. We write w as $w(t,x) = \int_0^t V(t-s,s,x)\,ds$, where, for each fixed $s \geqslant 0$, V solves

$$\begin{aligned} (\partial_{tt}^2 - \Delta)V(t,s,x) &= 0 \quad \text{in } Q, \\ \partial_\nu V(t,s,x) &= 0 \quad \text{on } \Sigma, \\ V(0,s,x) &= 0, \\ \partial_t V(0,s,x) &= G(s,x) \quad \text{in } \Omega. \end{aligned}$$

We have

$$\begin{aligned} &|w(t)|_{L^\infty(\Omega_R)} + |\partial_t w(t)|_{L^\infty(\Omega_R)} + |\nabla w(t)|_{L^\infty(\Omega_R)} \\ &\quad \leqslant \int_0^t |V(t-s,s,\cdot)|_{L^\infty(\Omega_R)} + |\partial_t V(t-s,s,\cdot)|_{L^\infty(\Omega_R)} \\ &\qquad + |\nabla V(t-s,s,\cdot)|_{L^\infty(\Omega_R)}\,ds. \end{aligned}$$

The thesis follows from application of (5). $\square$

At last we consider the nonhomogeneous initial boundary value problem

$$\begin{aligned} (\partial_{tt}^2 - \Delta)w &= G \quad \text{in } Q, \\ \partial_\nu w &= 0 \quad \text{on } \Sigma, \\ w(0,x) &= w_0, \\ \partial_t w(0,x) &= w_1 \quad \text{in } \Omega. \end{aligned} \tag{18}$$

From Lemmas 2.2 and 2.3, by linearity we have

Corollary 2.1. *Let* (w_0, w_1) *have compact support and satisfy* $w_0 \in H^3, w_1 \in H^2$. *Let* $G(t,\cdot)$ *have compact support independent of* t *and satisfy* $G(t,\cdot) \in H^2$ *for each* $t > 0$. *Then the solution* w *of* (18) *satisfies the estimate*

$$\begin{aligned} &|w(t)|_{L^\infty(\Omega_R)} + |\partial_t w(t)|_{L^\infty(\Omega_R)} + |\nabla w(t)|_{L^\infty(\Omega_R)} \\ &\quad \leqslant C_R(1+t)^{-1}(||w_0||_{H^3} + ||w_1||_{H^2}) \\ &\qquad + C_R \int_0^t (1+t-s)^{-1} ||G(s,\cdot)||_{H^2}\,ds \end{aligned} \tag{19}$$

for every $R > r_0$ *and* $t \geqslant 0$, *where* C_R *depends on* R, *the support of the data and the geometry of* $\partial\Omega$.

Similarly we can show the following result.

Lemma 2.4. *Let* (w_0, w_1) *and* $G(t,\cdot)$ *have compact support for each* $t>0$, *the support of* $G(t,\cdot)$ *being independent of* t. *If* $(w_0, w_1)\in H^3\times H^2$, $G(t,\cdot)\in H^2$ *for each* $t>0$, *then the solution* w *of* (18) *satisfies the estimate*

$$\|w(t)\|_{H^3(\Omega_R)} \leqslant C_R(1+t)^{-1}(\|w_0\|_{H^3}+\|w_1\|_{H^2}) + C_R\int_0^t (1+t-s)^{-1}\|G(s,\cdot)\|_{H^2}\,ds. \tag{20}$$

The above inequalities hold for every $R>r_0$ *and* $t\geqslant 0$; C_R *depends on* R, *the support of the data and the geometry of* $\partial\Omega$.

3. Proof of Theorem 1.1

Let us take functions $\tilde f, \tilde g: \mathbf{R}^2\to\mathbf{R}$ such that $\tilde f = f$, $\tilde g = g$ on Ω, and such that $\tilde f\in W^{6,1}(\mathbf{R}^2)$, $\tilde g\in W^{5,1}(\mathbf{R}^2)$,

$$\|\tilde f\|_{W^{6,1}(R^2)} + \|\tilde g\|_{W^{5,1}(R^2)} \leqslant C(\|f\|_{W^{6,1}} + \|g\|_{W^{5,1}}).$$

Let u_1 be the solution of the Cauchy problem

$$\begin{aligned} (\partial_{tt}^2-\Delta)u_1 &= 0 \quad \text{in } [0,\infty)\times\mathbf{R}^2,\\ u_1(0,x) &= \tilde f(x),\\ \partial_t u_1(0,x) &= \tilde g(x) \quad \text{in } \mathbf{R}^2. \end{aligned} \tag{21}$$

From [9, Theorem 2.1], we have

$$|\partial_t u_1(t)|_{L^\infty(R^2)} + |\nabla u_1(t)|_{L^\infty(R^2)} \leqslant C(1+t)^{-1/2}\|(\nabla\tilde f, \tilde g)\|_{W^{2,1}(R^2)}. \tag{22}$$

Choosing $r>r_0$ and $\chi(x)\in C_0^\infty(\mathbf{R}^2)$ so that $\chi(x)=1$ if $|x|\leqslant r$ and $=0$ if $|x|\geqslant r+1$, we put

$$u_2 = u-(1-\chi)u_1, \quad G = -u_1\Delta\chi - 2\nabla u_1\cdot\nabla\chi.$$

The function u_2 is the solution of the initial boundary value problem

$$\begin{aligned} (\partial_{tt}^2-\Delta)u_2 &= G \quad \text{in } Q,\\ \partial_\nu u_2 &= 0 \quad \text{on } \Sigma,\\ u_2(0,x) &= \chi f(x),\\ \partial_t u_2(0,x) &= \chi g(x) \quad \text{in } \Omega. \end{aligned} \tag{23}$$

Observe that $\operatorname{supp} G(t,\cdot) \subseteq \{x \mid r \leqslant |x| \leqslant r+1\}$ for all $t \geqslant 0$, and $\operatorname{supp} \chi f \subseteq \Omega_{r+1}$, $\operatorname{supp} \chi g \subseteq \Omega_{r+1}$.

Lemma 3.1. *The solution u_2 of* (23) *satisfies the estimate*

$$\begin{aligned} &|u_2(t)|_{L^\infty(\Omega_{r+2})} + |\partial_t u_2(t)|_{L^\infty(\Omega_{r+2})} + |\nabla u_2(t)|_{L^\infty(\Omega_{r+2})} \\ &\quad \leqslant C_r M_1 (1+t)^{-1/2} \log(e+t) \end{aligned} \tag{24}$$

for every $t \geqslant 0$, where $M_1 = ||f||_{W^{5,1}} + ||g||_{W^{4,1}}$.

Proof. From (19) with $w = u_2, w_0 = \chi f, w_1 = \chi g$, we obtain

$$\begin{aligned} &|u_2(t)|_{L^\infty(\Omega_{r+2})} + |\partial_t u_2(t)|_{L^\infty(\Omega_{r+2})} + |\nabla u_2(t)|_{L^\infty(\Omega_{r+2})} \\ &\quad \leqslant C_r (1+t)^{-1} (||\chi f||_{H^3} + ||\chi g||_{H^2}) \\ &\qquad + C_r \int_0^t (1+t-s)^{-1} ||G(s)||_{H^2}\, ds. \end{aligned} \tag{25}$$

We estimate $||G(s)||_{H^2}$. First of all we observe that

$$||G(s)||_{H^2} \leqslant C_r \sum_{|\alpha| \leqslant 3} |\partial^\alpha u_1(s)|_{L^\infty(\Omega_{r+1})}.$$

The solution u_1 is estimated by the $L^1 - L^\infty$ decay estimate (cf. [4])

$$|u_1(s,\cdot)|_{L^\infty} \leqslant C(1+s)^{-1/2} (||\tilde{f}||_{W^{2,1}} + ||\tilde{g}||_{W^{1,1}}). \tag{26}$$

To complete the estimate of $||G(s)||_{H^2}$, we apply (22) to $\partial^\alpha u_1(s)$, $|\alpha| \leqslant 2$, in order to obtain

$$\sum_{1 \leqslant |\alpha| \leqslant 3} |\partial^\alpha u_1(s)|_{L^\infty(\Omega_{r+1})} \leqslant C(1+s)^{-1/2} ||(\nabla f, g)||_{W^{4,1}}. \tag{27}$$

Thus, from (26) and (27) we get

$$||G(s)||_{H^2} \leqslant C_r M_1 (1+s)^{-1/2}. \tag{28}$$

From (25), (28) and (A.1) in the appendix we obtain

$$
\begin{aligned}
&|u_2(t)|_{L^\infty(\Omega_{r+2})} + |\partial_t u_2(t)|_{L^\infty(\Omega_{r+2})} + |\nabla u_2(t)|_{L^\infty(\Omega_{r+2})} \\
&\quad \leqslant C_r(1+t)^{-1}(||\chi f||_{H^3} + ||\chi g||_{H^2}) \\
&\qquad + C_r M_1 \int_0^t (1+t-s)^{-1}(1+s)^{-1/2}\, ds \\
&\quad \leqslant C_r M_1 (1+t)^{-1/2} \log(e+t) \quad \forall t \geqslant 0.
\end{aligned} \tag{29}
$$

Observe that by a Sobolev imbedding theorem $||f||_{H^3} \leqslant C||f||_{W^{4,1}}, ||g||_{H^2} \leqslant C||g||_{W^{3,1}}$. □

Choosing $\psi(x) \in C_0^\infty(\mathbf{R}^2)$ so that $\psi(x) = 1$ if $|x| \geqslant r+2$ and $= 0$ if $|x| \leqslant r+1$, we observe that

$$
\psi\chi f = 0, \quad \psi\chi g = 0, \quad \psi G = 0.
$$

Let us define

$$
H = -u_2 \Delta\psi - 2\nabla u_2 \cdot \nabla\psi.
$$

Observe that $\operatorname{supp} H(t,\cdot) \subseteq \{x \mid r+1 \leqslant |x| \leqslant r+2\}$ for all $t \geqslant 0$. The function ψu_2 solves the Cauchy problem

$$
\begin{aligned}
(\partial_{tt}^2 - \Delta)(\psi u_2) &= H \quad \text{in } [0,\infty) \times \mathbf{R}^2, \\
\psi u_2(0,x) &= 0, \\
\partial_t(\psi u_2)(0,x) &= 0 \quad \text{in } \mathbf{R}^2.
\end{aligned} \tag{30}
$$

Lemma 3.2. *The solution ψu_2 of* (30) *satisfies the estimate*

$$
|\partial_t(\psi u_2)(t)|_{L^\infty(R^2)} + |\nabla(\psi u_2)(t)|_{L^\infty(R^2)} \leqslant C_r M_2 (1+t)^{-1/2} \log^2(e+t) \tag{31}
$$

for every $t \geqslant 0$, where $M_2 = ||f||_{W^{6,1}} + ||g||_{W^{5,1}}$.

Proof. Without loss of generality we may assume that t is large, say $t \geqslant r$. First, let us assume $t/N \leqslant |x| \leqslant t$, where $N > 1$ will be fixed later on. By the Duhamel's principle we can write $\psi u_2(t,x) = \int_0^t U(t-s,s,x)\, ds$, where, for each fixed $s \geqslant 0$, U solves

$$
\begin{aligned}
(\partial_{tt}^2 - \Delta) U(t,s,x) &= 0 \quad \text{in } [0,\infty) \times \mathbf{R}^2, \\
U(0,s,x) &= 0, \\
\partial_t U(0,s,x) &= H(s,x) \quad \text{in } \mathbf{R}^2.
\end{aligned}
$$

We recall Klainerman's inequality [5] in the plane

$$|v(t,x)| \leqslant C(1+t+|x|)^{-1/2}(1+|t-|x||)^{-1/2}|||v(t)|||_2 \quad \forall t \geqslant 0 \tag{32}$$

which holds for all smooth functions vanishing sufficiently rapidly as $|x| \to \infty$, so that the norm in the right side is finite for each fixed $t \geqslant 0$. Applying (32) gives

$$|\partial_t U(t,s,x)| \leqslant C(1+t+|x|)^{-1/2}(1+|t-|x||)^{-1/2}|||\partial_t U(t,s,\cdot)|||_2,$$

$$|\nabla U(t,s,x)| \leqslant C(1+t+|x|)^{-1/2}(1+|t-|x||)^{-1/2}|||\nabla U(t,s,\cdot)|||_2. \tag{33}$$

The commutation rules of the generalized derivatives Γ_i with the wave operator and an energy argument give for every multi-index A

$$||\partial_t \Gamma^A U(t,s,\cdot)||^2 + ||\nabla \Gamma^A U(t,s,\cdot)||^2 = ||\partial_t \Gamma^A U(0,s,\cdot)||^2 + ||\nabla \Gamma^A U(0,s,\cdot)||^2$$

for every $t, s \geqslant 0$, where in the right side the norms are evaluated at time $t = 0$. It readily follows that

$$\begin{aligned}
&|||\partial_t U(t,s,\cdot)|||_2 + |||\nabla U(t,s,\cdot)|||_2 \\
&\quad \leqslant C \max_{|A| \leqslant 2} ||\Gamma^A D U(t,s,\cdot)|| \\
&\quad \leqslant C \max_{|A| \leqslant 2} \left(||D\Gamma^A U(t,s,\cdot)|| + \sum_{|B| \leqslant |A|-1} |\delta_{AB}| \, ||D\Gamma^B U(t,s,\cdot)|| \right) \\
&\quad \leqslant C \max_{|A| \leqslant 2} \left(||\partial_t \Gamma^A U(0,s,\cdot)|| + ||\nabla \Gamma^A U(0,s,\cdot)|| \right) \\
&\quad \leqslant C ||H(s,\cdot)||_{H^2(\Omega_{r+2})} \leqslant C ||u_2(s,\cdot)||_{H^3(\Omega_{r+2})}
\end{aligned} \tag{34}$$

for every $t, s \geqslant 0$. On the other hand, applying (20) and (28) to the solution u_2 of (23) yields

$$\begin{aligned}
||u_2(t,\cdot)||_{H^3(\Omega_{r+2})} &\leqslant C_r M_1 (1+t)^{-1} \\
&\qquad + C_r M_1 \int_0^t (1+t-s)^{-1}(1+s)^{-1/2} \, ds \\
&\leqslant C_r M_1 (1+t)^{-1/2} \log(e+t) \quad \forall t \geqslant 0.
\end{aligned} \tag{35}$$

Then, from (33)–(35) one has

$$\begin{aligned}
&|\partial_t(\psi u_2)(t,x)| + |\nabla(\psi u_2)(t,x)| \\
&\quad \leqslant C_r M_1 \int_0^t (1+t-s+|x|)^{-1/2} \\
&\qquad \times (1+|t-s-|x||)^{-1/2}(1+s)^{-1/2} \log(e+s) \, ds.
\end{aligned} \tag{36}$$

Recalling that $t/N \leqslant |x| \leqslant t$, we decompose the integral by $\int_0^t = \int_0^{t-|x|} + \int_{t-|x|}^t$. We obtain

$$\begin{aligned}
&\int_0^{t-|x|} (1+t-s+|x|)^{-1/2}(1+t-s-|x|)^{-1/2}(1+s)^{-1/2}\log(e+s)\,ds \\
&\leqslant (1+2t/N)^{-1/2}\log(e+t)\int_0^{t-|x|}(t-s-|x|)^{-1/2}s^{-1/2}\,ds \\
&= (1+2t/N)^{-1/2}\log(e+t)\int_0^1 (1-\sigma)^{-1/2}\sigma^{-1/2}\,ds \\
&= B(1/2,1/2)(1+t/N)^{-1/2}\log(e+t),
\end{aligned} \tag{37}$$

where we have used the change of variables $s = (t-|x|)\sigma$ and where B denotes the Euler Beta function. Moreover,

$$\begin{aligned}
&\int_{t-|x|}^t (1+t-s+|x|)^{-1/2}(1+|t-s-|x||)^{-1/2}(1+s)^{-1/2}\log(e+s)\,ds \\
&\leqslant (1+t/N)^{-1/2}\log(e+t)\int_{t-|x|}^t (1+s-t+|x|)^{-1/2}(1+s)^{-1/2}\,ds \\
&= (1+t/N)^{-1/2}\log(e+t)\int_0^{|x|}(1+\tau)^{-1/2}(1+t-|x|+\tau)^{-1/2}\,d\tau \\
&\leqslant (1+t/N)^{-1/2}\log(e+t)\int_0^{|x|}(1+\tau)^{-1}\,d\tau \\
&= (1+t/N)^{-1/2}\log(e+t)\log(1+|x|) \\
&\leqslant (1+t/N)^{-1/2}\log^2(1+t),
\end{aligned} \tag{38}$$

where we have used the change of variables $\tau = s - t + |x|$. We obtain from (36)–(38)

$$|\partial_t(\psi u_2)(t,x)| + |\nabla(\psi u_2)(t,x)| \leqslant C_r M_1 (1+t/N)^{-1/2}\log^2(e+t). \tag{39}$$

Next, let us consider the case $r+3 \leqslant |x| \leqslant t/N$. By the Poisson formula (cf. [2])

$$U(t,s,x) = \frac{1}{2\pi}\int_{|x-y|<t}\frac{H(s,y)}{\sqrt{t^2-|x-y|^2}}\,dy.$$

By the change of variables $y = x+\rho\omega,\ 0 \leqslant \rho < t, \omega \in S^1$, and an integration by parts, we obtain

$$U(t,s,x) = tH(s,x) + \frac{1}{2\pi}\int_0^t\int_{S^1}\sqrt{t^2-\rho^2}\,\frac{\partial H}{\partial \rho}(s,x+\rho\omega)\,d\rho\,d\omega.$$

The first term on the right side is zero because $r+2<|x|$. Differentiation in time and the inverse change of variables yields

$$|\partial_t U(t,s,x)| \leqslant \frac{1}{2\pi}\int_{|x-y|<t} \frac{|\nabla H(s,y)|}{\sqrt{t^2-|x-y|^2}}\,dy.$$

For $|\nabla U(t,s,x)|$ we can get the same estimate. Then we have

$$|\partial_t \psi u_2(t,x)| + |\nabla \psi u_2(t,x)| \leqslant \frac{1}{\pi}\int_0^t \int_{|x-y|<t-s} \frac{|\nabla H(s,y)|}{\sqrt{(t-s)^2-|x-y|^2}}\,dy\,ds.$$

As in (35) we obtain

$$|\nabla H(s,\cdot)|_\infty \leqslant ||u_2(s,\cdot)||_{H^4(\Omega_{r+2})} \leqslant C_r M_2 (1+s)^{-1/2}\log(e+s) \quad \forall s \geqslant 0,$$

where $M_2 = ||f||_{W^{6,1}} + ||g||_{W^{5,1}} \geqslant M_1$. Recalling that $\operatorname{supp} H(s,\cdot) \subseteq \{|y| \leqslant r+2\}$ we obtain

$$\begin{aligned} &|\partial_t \psi u_2(t,x)| + |\nabla \psi u_2(t,x)| \\ &\quad \leqslant C_r M_2 \int_0^t \int_{|x-y|<t-s,|y|\leqslant r+2} \frac{(1+s)^{-1/2}\log(e+s)}{\sqrt{(t-s)^2-|x-y|^2}}\,dy\,ds. \end{aligned} \tag{40}$$

We decompose the integral by $\int_0^t = \int_0^{t/2} + \int_{t/2}^t = I_1 + I_2$. We have

$$\begin{aligned} I_1 &\leqslant \frac{\log(e+t)}{\sqrt{t/2}} \int_0^{t/2} \int_{|x-y|<t-s,|y|\leqslant r+2} \frac{1}{\sqrt{s}\sqrt{t-s-|x-y|}}\,dy\,ds \\ &\leqslant \frac{\log(e+t)}{\sqrt{t/2}} \int_0^{t} \int_{|x-y|<t-s,|y|\leqslant r+2} \frac{1}{\sqrt{s}\sqrt{t-s-|x-y|}}\,dy\,ds \\ &= \frac{\log(e+t)}{\sqrt{t/2}} \int_{|x-y|<t,|y|\leqslant r+2} \int_0^{t-|x-y|} \frac{1}{\sqrt{s}\sqrt{t-|x-y|-s}}\,ds\,dy. \end{aligned}$$

By the change of variables $s=(t-|x-y|)\sigma$ in the last integral, we obtain

$$I_1 \leqslant \frac{\log(e+t)}{\sqrt{t/2}} \int_{|y|\leqslant r+2} \int_0^1 \sigma^{-1/2}(1-\sigma)^{-1/2}\,d\sigma\,dy \leqslant C_r t^{-1/2}\log(e+t). \tag{41}$$

As regards I_2, let us assume t and N so large that $t(1/2 - 1/N) > r + 2$, which yields $t - |x - y| > t/2$. Then

$$
\begin{aligned}
I_2 &\leqslant (1 + t/2)^{-1/2} \log(e + t) \int_{t/2}^{t} \int_{|x-y| < t-s, |y| \leqslant r+2} \frac{1}{\sqrt{(t-s)^2 - |x-y|^2}} \, dy \, ds \\
&\leqslant (1 + t/2)^{-1/2} \log(e + t) \int_{|x-y| < t/2, |y| \leqslant r+2} \int_{t/2}^{t-|x-y|} \frac{1}{\sqrt{(t-s)^2 - |x-y|^2}} \, ds \, dy.
\end{aligned}
$$

Consider the change of variable $\tau = t - |x - y| - s$. Then $t + |x - y| - s = 2|x - y| + \tau \geqslant 2|x| - 2(r + 2) + \tau \geqslant 2 + \tau$. It follows that

$$
\begin{aligned}
I_2 &\leqslant (1 + t/2)^{-1/2} \log(e + t) \int_{|x-y| < t/2, |y| \leqslant r+2} \int_{0}^{t/2-|x-y|} \frac{1}{\sqrt{\tau(2+\tau)}} \, d\tau \, dy \\
&\leqslant (1 + t/2)^{-1/2} \log(e + t) \int_{|y| \leqslant r+2} \int_{0}^{t} \frac{1}{\sqrt{\tau(2+\tau)}} \, d\tau \, dy \\
&\leqslant C(1 + t)^{-1/2} \log(e + t) \left(\int_0^1 \frac{1}{\sqrt{\tau}} \, d\tau + \int_1^t \frac{d\tau}{\tau} \right) \\
&\leqslant C(1 + t)^{-1/2} \log^2(e + t). \qquad (42)
\end{aligned}
$$

From (40)–(42) we have

$$
|\partial_t \psi u_2(t, x)| + |\nabla \psi u_2(t, x)| \leqslant C_r M_2 (1 + t)^{-1/2} \log^2(e + t). \qquad (43)
$$

Finally, let us consider the case $|x| \leqslant r + 3 \leqslant t/N$. Since $\partial_\nu(\psi u_2) = 0$ on Σ, ψu_2 solves the initial boundary value problem (23) with H instead of G and zero initial data. Applying the local decay estimate (19) gives

$$
\begin{aligned}
&|\partial_t \psi u_2(t)|_{L^\infty(\Omega_{r+3})} + |\nabla \psi u_2(t)|_{L^\infty(\Omega_{r+3})} \\
&\quad \leqslant C_r \int_0^t (1 + t - s)^{-1} ||H(s)||_{H^2} \, ds \\
&\quad \leqslant C_r M_1 \int_0^t (1 + t - s)^{-1} (1 + s)^{-1/2} \, ds \\
&\quad \leqslant C_r M_1 (1 + t)^{-1/2} \log(e + t). \qquad (44)
\end{aligned}
$$

Combining (39), (43) and (44) implies the lemma. $\square$

Since $u = (1-\chi)u_1 + u_2$, we have

$$\begin{aligned}
&|\partial_t u(t)|_\infty + |\nabla u(t)|_\infty \\
&\quad \leqslant |(1-\chi)\partial_t u_1(t)|_\infty + |\nabla((1-\chi)u_1(t))|_\infty + |\partial_t u_2(t)|_\infty + |\nabla u_2(t)|_\infty \\
&\quad \leqslant |\partial_t u_1(t)|_{L^\infty(R^2)} + |\nabla u_1(t)|_{L^\infty(R^2)} + C|u_1(t)|_{L^\infty(R^2)} \\
&\qquad + |\partial_t u_2(t)|_{L^\infty(\Omega_{r+2})} + |\nabla u_2(t)|_{L^\infty(\Omega_{r+2})} + C|u_2(t)|_{L^\infty(\Omega_{r+2})} \\
&\qquad + |\partial_t(\psi u_2(t))|_{L^\infty(R^2)} + |\nabla(\psi u_2(t))|_{L^\infty(R^2)}.
\end{aligned}$$

From (22), (24), (26) and (31) we finally obtain

$$\begin{aligned}
&|\partial_t u(t)|_\infty + |\nabla u(t)|_\infty \\
&\quad \leqslant CM_1(1+t)^{-1/2}\log(e+t) + C_r M_2(1+t)^{-1/2}\log^2(e+t) \\
&\quad \leqslant C_r M_2(1+t)^{-1/2}\log^2(e+t) \quad \forall t \geqslant 0. \qquad (45)
\end{aligned}$$

This estimate gives the required decay rate.

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Appendix

We report the elementary estimate used above.

Lemma A.1. *There exists a constant $C > 0$ such that for all $t \geqslant 0$*

$$\int_0^t (1+t-s)^{-1}(1+s)^{-1/2}\, ds \leqslant C(1+t)^{-1/2}\log(1+t). \qquad (A.1)$$

Proof. The estimate may be proved following the lines of the proof of (5.49) in [7]. $\square$

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